Mod-2

//-> **Numerical differentiat^n:-** It's the process for determining the value of derivative of a funct^n at some values of the arguments from the given set of values of funct^n.

To determine the values of the derivatives of $y = f(x)$, we've to approx. the funct^n $y = f(x)$ by an interpolating polynomial & then differentiate the funct^n as many times

When a funct^n $y = f(x)$ is defined at a discrete set of pts. as $x_0, x_1, x_2, \cdots, x_n$ $\Rightarrow f_0, f_1, f_2, \cdots, f_n$, where $f_i = f(x_i)$.

Determinat^n of the derivative of the funct^n $f(x)$ as the limit is given by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$, where, h is a very small +ve integer.

Whenever, it's complicated to find the derivative of a funct^n as a limit of quotient or become too tedious & difficult, we use another alternative called Numerical differentiat^n.

//-> **Simple numerical method for derivative:-**

We know, $$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$= \lim_{h \to 0} \frac{\Delta f(x_0)}{h}.$$

$$\Rightarrow \boxed{f'(x_0) \simeq \frac{\Delta f(x_0)}{h}}$$ → Now, this is called the forward diff. formula for derivative.

Expanding $f(x+h)$ in Taylor's Series expan^n, we've:

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots$$

$$\Rightarrow \boxed{f(x+h) = \sum_{i=0}^{n} \frac{h^i f^i(x)}{i!}}$$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} = f'(x) + \frac{h}{2!} f''(x) + \frac{h^2}{3!} f'''(x) +$$

$$\Rightarrow f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{h}{2!} f''(x) - \frac{h^2}{3!} f'''(x) -$$

$$\Rightarrow f'(x) \simeq \frac{\Delta f(x)}{h} + T(h)$$

where, $T(h) = -\frac{h}{2!} f''(x) - \frac{h^2}{3!} f'''(x) - \cdots$

denotes the truncat^n error in the approximat^n of $\frac{\Delta f(x)}{h}$ of $f'(x)$.

(IMP) The difference quotient $\frac{f(x+h) - f(x)}{h}$ is an approximat^n of the derivative $f'(x)$ & this approx. gets better as (h) gets smaller.

N.B: Taylor's Th^m with remainder gives the taylor series, then:

$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(\xi)$, where $\xi \in (x-h, x+h)$

Rearranging, we get $f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{h}{2!} f''(\xi)$, which tells us that the error is equal to (h). So, T(h) = O(h), where O(h) is order of (h), & thus, the approx. $\frac{\Delta f(x)}{h}$ is of order O(h), i.e. this method is said to be of 1st order approx.

Q: Find approx. to $\frac{d}{dx}(x^{-1})$ at $x = 1$ for $h = 0.2$, $h = 0.02$; $h = 0.002$

$$\boxed{\text{N.B} \Rightarrow f'(x) = \frac{\Delta f(x)}{h} = \frac{f(x+h) - f(x)}{h}}$$

Ans - $f(x) = x^{-1} = \frac{1}{x}$

$$\Rightarrow f'(x) = \frac{\Delta f(x)}{h} \Rightarrow f'(1) = \frac{\Delta f(1)}{h}$$

$$= \frac{f(1+h) - f(1)}{h}$$

$$\frac{\frac{1}{1+h} - 1}{h} = \frac{\cancel{1} - \cancel{1} - h}{h(1+h)} = \frac{-1}{1+h}$$

∴

h	$f'(1)$
0.2	−0.8333
0.02	−0.9803
0.002	−0.99800

II → Central diff. formula of order $O(h^2)$:-

Let (f) is continuously differentiable in the neighbourhood of (x) i.e. $(x-h, x+h)$. Applying Lagrange's Mean Value Thm to (f), we get $\frac{f(x+h) - f(x-h)}{(x+h) - (x-h)} = f'(\xi)$, where $\xi \in (x-h, x+h)$

$$\therefore \frac{f(x+h) - f(x-h)}{2h} = f'(\xi)$$

Taking $\lim_{h \to 0}$ on both sides, we get:

$$\lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h} = \lim_{h \to 0} f'(\xi)$$

$$= f'(x)$$

$$\Rightarrow \boxed{f'(x) \simeq \frac{f(x+h) - f(x-h)}{2h} = \frac{\delta f(x)}{2h}} \quad \text{---} ①$$

Here, $f(x+h) - f(x-h) = \delta f(x)$ is called central diff. of $f(x)$ & eqn. ① is called the central diff. formula for numerical differentiatⁿ.

Now, expanding $f(x+h)$ & $f(x-h)$ in Taylor's series, we've:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2 f''(x)}{2!} + \frac{h^3 f'''(x)}{3!} + \cdots$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2 f''(x)}{2!} - \frac{h^3 f'''(x)}{3!} + \cdots$$

$$- \qquad - \qquad + \qquad - \qquad +$$

$$f(x+h) - f(x-h) = 2\left[hf'(x) + \frac{h^3 f''(x)}{3!} + \cdots\right]$$

$$\Rightarrow \frac{f(x+h) - f(x-h)}{2h} = \left[f'(x) + \frac{h^2 f'''(x)}{3!} + \cdots\right]$$

$$\Rightarrow f'(x) \simeq \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{3!} f'''(x) - \frac{h^4}{5!} f'''''(x) - \cdots$$

$$= \frac{\delta f(x)}{2h} - \frac{h^2}{3!} f'''(\xi), \ \xi \in (x-h, x+h)$$

Hence, for small value of (h), the truncatⁿ error in central diff. formula is: $T(h) \doteq -\frac{h^2 f'''(\xi)}{3!}$. So, the central diff. formula is of order $O(h^2)$

Q: Find $\frac{d}{dx}(x^{-1})$ by using CDF at $x = 1$ & for $h = 0.2, 0.02, 0.002$.

Ans- $f(x) = x^{-1} = \frac{1}{x}$

$$\Rightarrow f'(x) = \frac{\delta f(x)}{2h} = \frac{f(x+h) - f(x-h)}{2h}$$

$$= \frac{\frac{1}{x+h} - \frac{1}{x-h}}{2h}$$

$$= \frac{-1}{x-h^2}$$

$$\boxed{N.B \Rightarrow f'(x) = \frac{\delta f(x)}{2h} = \frac{f(x+h) - f(x-h)}{2h}}$$

$$\therefore f'(1) = \frac{-1}{1-h^2}$$

$\therefore$

h	$f'(1)$
0.2	−1.041
0.02	−1.000400
0.002	−1.000004

Q: Find $\frac{d}{dx}(\sqrt{x})$ at $x = 1$ by using CDF with $h = 0.1, 0.01, 0.001$

Ans.- $f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{\delta f(x)}{2h}$

$$= \frac{f(x+h) - f(x-h)}{2h}$$

$$= \frac{\sqrt{x+h} - \sqrt{x-h}}{2h}$$

$$\therefore f'(1) = \frac{\sqrt{1+h} - \sqrt{1-h}}{2h}$$

$\therefore$

h	$f'(1)$
0.1	0.5006
0.01	0.500006
0.001	0.5000000625

11→ Interpolatory formula:- Let's consider $(n+1)$ distinct nodes $x_0, x_1, \ldots, x_n$ in the interval $[a,b]$ & let (f) be a continuously differentiable funct^n in $[a,b]$. Now, the Lagrange's Interpolating polynomial can be written as:

$$f(x) \simeq P(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + \dots + l_n(x) f(x_n)$$

$$\therefore f(x) = P_n(x) + E(x)$$

$$= \sum_{i=0}^{n} l_i(x) f(x_i) + \frac{(x-x_0)\dots(x-x_n)}{(n+1)!} \cdot f^{n+1}(\eta)$$

where, $\eta \in (x-h, x+h)$

Differentiating both sides, we get:

$$f'(x) = \sum_{i=0}^{n} l_i'(x) f(x_i) + \frac{d}{dx}\left[\frac{(x-x_0)\dots(x-x_n)}{(n+1)!} f^{n+1}(\eta)\right]$$

Now, ignoring the error term, we obtain

$$f'(x) = l_0'(x) f(x_0) + l_1'(x) f(x_1) + \dots + l_n'(x) f(x_n)$$

is called an $(n+1)$ pt. formula for the approximatⁿ of derivative of $f(x)$.

II → Special Case :-

When $n=2$: Let the nodes are x_0, x_1 & x_2.

$$\therefore l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$\therefore l_0'(x) = \frac{2x-x_2-x_1}{(x_0-x_1)(x_0-x_2)} \quad ; \quad l_1'(x) = \frac{2x-x_2-x_0}{(x_1-x_0)(x_1-x_2)}$$

$$l_2'(x) = \frac{2x-x_1-x_0}{(x_2-x_0)(x_2-x_1)}$$

$$\Rightarrow f'(x) = \frac{2x-x_2-x_1}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{2x-x_2-x_0}{(x_1-x_0)(x_1-x_2)} f(x_1)$$

$$+\frac{2x-x_1-x_0}{(x_2-x_0)(x_2-x_1)}f(x_2)$$

Further, if the nodes in the above case are equispaced by step length (h), then:

- $f'(x_0) = -\frac{3}{2h}f(x_0) + \frac{2}{h}f(x_1) - \frac{1}{2h}f(x_2)$

$$= \frac{1}{h}\left[-\frac{3}{2}f(x_0) + 2f(x_0+h) - \frac{1}{2}f(x_0+2h)\right]$$

- $f'(x_1) = \frac{1}{h}\left[-\frac{1}{2}f(x_0) + \frac{1}{2}f(x_1)\right]$

- $f'(x_2) = \frac{1}{h}\left[\frac{1}{2}f(x_0) - 2f(x_1) + \frac{3}{2}f(x_2)\right]$

→ Approximatⁿ formulas for higher order derivatives:-

we know, $f'(x) = \frac{\Delta f(x)}{h}$

$\therefore f''(x) = \frac{\Delta f'(x)}{h} = \frac{\Delta\Delta f(x)}{h^2} = \frac{\Delta^2 f(x)}{h^2}$

$\therefore \boxed{f^k(x) = \frac{\Delta^k f(x)}{h^k}}$, which is the approximatⁿ formula for the evaluatⁿ of kth order derivative of the functⁿ $f(x)$.

In similar manner, we can derive the formula $\boxed{f^k(x) = \frac{\nabla^k f(x)}{h^{(k)}}}$ starting from $f'(x) = \frac{\nabla f(x)}{h}$. $\boxed{\text{This } f^k(x) = \frac{\nabla^k f(x)}{h^{(k)}}}$ is called the backward diff. approximatⁿ to the kth order derivative of $f(x)$

→ CDF for higher order derivative:-

we know $f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$

Math

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f'''(x) + \cdots$$

Now, ~~subtracting~~ adding both the equ^{n}s above, we've:

$$f(x+h) + f(x-h) = 2\left[f(x) + \frac{h^2}{2!} f''(x) + \frac{h^4}{4!} f'''(\xi)\right]$$

$$\Rightarrow f''(x) = \frac{f(x-h) - 2f(x) + f(x+h)}{h^2} - \frac{2h^4}{4!} f(\xi)$$

$$\simeq \frac{\delta^2 f(x)}{h^2} - \frac{2h^4}{4!} f(\xi)$$

Hence, the higher order derivatives using CDF can be found by $\boxed{f^{(k)}(x) = \frac{\delta^k f(x)}{h^k}}$

Q: For $f(x) = \ln x$, find the approximat^n of $f''(1)$ using CDF with $h = 0.1, 0.01, 0.001$

Ans- $f(x) = \ln x$

$$\therefore \boxed{f''(x) = \frac{f(x-h) - 2f(x) + f(x+h)}{h^2}}$$

$$= \frac{\ln(x-h) - 2\ln x + \ln(x+h)}{h^2}$$

$$\therefore f''(1) = \frac{\ln(1-h) - 2\ln 1 + \ln(1+h)}{h^2}$$

∴

h	$f''(1)$
0.1	−1.005
0.01	−1.00005
0.001	−1.0000005

// -> Numerical integrat^n:-

⊛ It's also known as "numerical quadrature" or "mechanical quadrature" or "approximate evaluat^n of integrals"

⊛ To integrate $I(f) = \int_a^b f(x)\,dx$ numerically we've to find the approximating polynomial $P(x)$ such that $f(x) \simeq P(x)$. where, $f(x) \in [a,b]$. So, $I(P) = \int_a^b P(x)dx$.

Now, according to weirstrass approximatⁿ thᵐ, we know that for a given +ve no. (ϵ) $\exists$ a polynomial $P(x)$ such that:

$\boxed{|f(x) - P(x)| < \epsilon/(b-a)}$, $x \in [a,b]$, where (ϵ) is the error tolerance. So:

$$|I(f) - I(P)| = \left|\int_a^b [f(x) - p(x)]dx\right|$$
$$\leq \int_a^b |f(x) - p(x)|dx < \epsilon$$

④ There are 2 methods to evaluate numerical quadrature. They're:

→ ⓐ Newton-Cotes method.

→ ⓑ Gauss method.

N.B: Riemann Sum

$\boxed{S = a_1 f(x_1) + a_2 f(x_2) + \dots + a_n f(x_n)}$ is a weighted sum & can be used as approximatⁿ to $I(f)$. Also, we can write it as: $\boxed{R_n(f) = a_1 f(x_1) + a_2 f(x_2) + \dots + a_n f(x_n)}$ & $R_n(f) \simeq I(f)$, where $R_n(f)$ is called n-pt. integratⁿ rule. $x_1, x_2, \dots, x_n$ are called nodes or abscissas & $a_1, a_2, \dots, a_n$ are called the weights or coefficients.

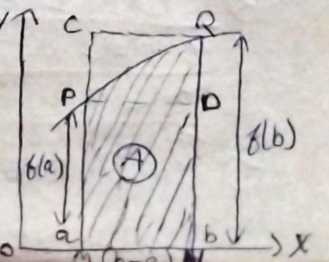
Now, $E(f) = I(f) - R_n(f)$, is called truncatⁿ error associated with the approximatⁿ $R_n(f)$ to $I(f)$.

II→ Some simple quadrature Rule:-

ⓐ Rectangular rule:-

Let $I(f) = \int_a^b f(x)dx$

● → The geometrical meaning



of $I(f)$ in the area A_1 bounded by curve $y=f(x)$, $x=a$, $x=b$.

→ when the pt. N→M, the area MNPD is approximately = to area A. Similarly, when M→N, the area of MNQC is nearly = to area A.

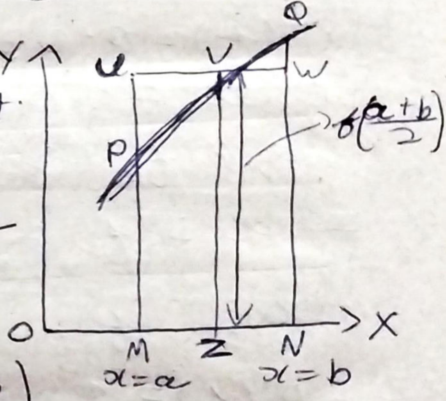
So, $\int_a^b f(x)dx = (b-a)\, f(a)$

or $\int_a^b f(x)dx = (b-a)\, f(b)$

$$\Rightarrow \left.\boxed{\begin{array}{l} R'(f) = (b-a)\, f(a) \\ R''(f) = (b-a)\, f(b) \end{array}}\right\} \text{----} \quad ①$$

which are respectively called as left & right rectangular rule.

ⓐ Mid-pt. rule:-



Here, Z is the mid-pt of MN.

So, area of MNWV is approximately = to area A. i.e.

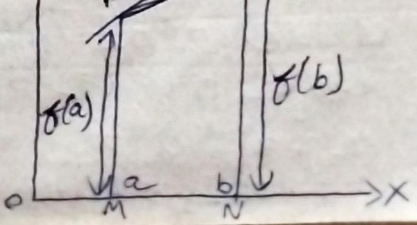
$\int_a^b f(x)dx = (b-a)\, f\left(\frac{a+b}{2}\right)$

$$\Rightarrow \boxed{M(f) = (b-a)\, f\left(\frac{a+b}{2}\right)} \quad \text{----} \quad ②$$

This is called mid-pt. quadrature rule.

ⓐ Trapezoidal rule:-

Here, the area of the trapezium MNPQ is approximately = to area A. So, area of trapezium = $\frac{1}{2} \times MN \times (MP + NQ)$



$= \frac{1}{2}(b-a)(f(a)+f(b))$

Thus, $\boxed{T(f) = \frac{1}{2}(b-a)\{f(a)+f(b)\}}$ is called ...... ⑥

trapezoidal rule.

N.B: $\boxed{T(f) = \frac{1}{2}[R'(f)+R''(f)]}$, i.e. T(f) is the mean of left & rt. rectangular rule.

Q: Find $I(f) = \int_{0.1}^{0.2} \sqrt{x}\,dx$ using $R'(f)$, $R''(f)$, $M(f)$, $T(f)$.

Ans- $R'(f) = 0.0316227766$

$R''(f) = 0.04472135955$

$M(f) = 0.0387298346$

$T(f) = 0.03817206808$

Exact $= 0.0385466$

Here, M(f) is more accurate.

Q: Find $I(f) = \int_{0}^{1} e^{-x^2}\,dx$ using $R'(f)$, $R''(f)$, $M(f)$, $T(f)$.

Ans- $R'(f) = 1$; $R''(f) = 0.36787944117$

$M(f) = 0.77880078307$

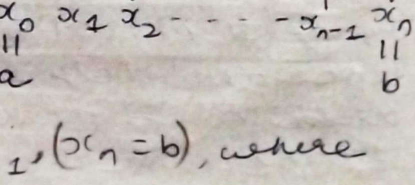
$T(f) = 0.68393972058$

Exact $= 0.74682413281$

Here, M(f) is more accurate.

// -> Newton-cotes rule:-

Let $I(f) = \int_a^b f(x)\,dx$



Divide [a, b] into (n) equal parts, i.e. $(x_0 = a), x_1, x_2, \ldots, x_{n-1}, (x_n = b)$, where $x_i - x_{i-1} = h$.

Now, to calculate $I(P)$, we use the Lagrange's IP.

$$\therefore P_n(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + \cdots + l_n(x)f(x_n)$$

If $f(x) \simeq P_n(x)$, $a \leq x \leq b$, then

$$I(f) = \int_a^b f(x)\,dx \simeq \int_a^b P_n(x)\,dx = I(P)$$

$$= f(x_0)\int_a^b l_0(x)\,dx + \cdots + f(x_n)\int_a^b l_n(x)\,dx$$

Let $\boxed{\omega_i = \int_a^b l_i(x)\,dx,\ i = 0(1)n}$ (ω = omega)

So, $\boxed{\begin{aligned} I(P_n) &= \omega_0 f_0 + \omega_1 f_1 + \cdots + \omega_n f_n \\ &= R_{n+1}(f) \end{aligned}}$ ---- ①

(This is Newton-cotes formula)

The rule $R_{n+1}(f)$ is called the $(n+1)$ pt. Newton-cotes quadrature formula involving $(n+1)$ nodes $x_0, x_1, \ldots, x_n$ & $(n+1)$ coefficients or wets: $\omega_0, \omega_1, \ldots, \omega_n$

$\Rightarrow$ To calculate $\omega_i = \int_a^b l_i(x)\,dx$

$$= \int_a^b \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}\,dx$$

Let $x_i = x_0 + ih$; $x = x_0 + \alpha h$; $0 \leq \alpha \leq n$

Let $dx = h\,d\alpha \Rightarrow$ when $x \to a$, $\alpha \to 0$

$x \to b$, $\alpha \to n$

$$\therefore \omega_i = \int_0^n \frac{\alpha h\cdot(\alpha-1)h\cdots(\alpha-(i-1))h(\alpha-(i+1))h\cdots(\alpha-n)h}{ih\cdot(i-1)h\cdots(i-(i-1))h(i-(i+1))h\cdots(i-n)h}(h\,d\alpha)$$

$$= \int_0^n \frac{\alpha(\alpha-1)\cdots(\alpha-i+1)(\alpha-i-1)\cdots(\alpha-n)}{i(i-1)\cdots(+1)(-1)\cdots(i-n)}\,h\,d\alpha$$

$$= \frac{h}{i!(-1)(-2)\cdots(i-n)}\int_0^n \alpha(\alpha-1)\cdots(\alpha-i+1)(\alpha-i-1)\cdots(\alpha-n)\,d\alpha$$

$$= \frac{h}{i!(-1)^{i-n}\, 1\cdot 2\cdot 3\cdots(n-i)} \int_0^n \alpha(\alpha-1)\cdots(\alpha-i+1)(\alpha-i-1)\cdots(\alpha-n)\,d\alpha$$

$$= \frac{h}{i!\,(-1)^{i-n}\,(n-i)!} \int_0^n \alpha(\alpha-1)\cdots(\alpha-i+1)(\alpha-i-1)\cdots(\alpha-n)\,d\alpha$$

$$= \boxed{\frac{h\cdot(-1)^{n-i}}{i!\,(n-i)!} \int_0^n \alpha(\alpha-1)\cdots(\alpha-i+1)(\alpha-i-1)\cdots(\alpha-n)\,d\alpha} \quad ---②$$

where $i = 0(1)n$

//-> Error in Newton-Cotes quadrature:-

We know $f(x) = P_n(x) + \frac{(x-x_0)\cdots(x-x_n)}{(n+1)!} f^{n+1}(\eta)$,

$\eta \in (a,b)$, for $f \in C^{n+1}[a,b]$. ~~$\eta \in (a,b)$~~

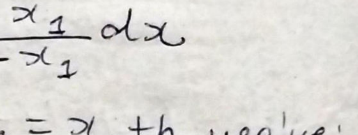
$\therefore \boxed{I(f) = R_{n+1}(f) + E(f)}$ So:

$$\boxed{E(f) = \frac{1}{(n+1)!} \int_a^b (x-x_0)\cdots(x-x_n) f^{n+1}(\eta)\,dx}$$

//-> (n = 1) Trapezoidal rule:-

Here, no. of node is 2 i.e. x_0, x_1

$\therefore x_0 = a,\ x_1 = b$



So, $I(P) = R_2(f) = w_0 f(a) + w_1 f(b)$

$\therefore w_0 = \int_a^b l_0(x)\,dx = \int_a^b \frac{x - x_1}{x_0 - x_1}\,dx$

Putting $x = x_0 + \alpha h$ & $x_1 = x_0 + h$, we've:

$w_0 = \int_0^1 \frac{(\alpha-1)h}{-h} h\,d\alpha \Rightarrow \boxed{w_0 = \frac{h}{2}}$

$\therefore w_1 = \int_a^b l_1(x)\,dx = \int_a^b \frac{x - x_0}{x_1 - x_0}\,dx$

$\Rightarrow \boxed{w_1 = h/2}$

$$\therefore I(P) = \frac{h}{2} f(a) + \frac{h}{2} f(b)$$

$$\Rightarrow I(P) = \frac{h}{2}[f(a)+f(b)] = \frac{b-a}{2}[f(a)+f(b)]$$

So, $\boxed{I(P) = \frac{b-a}{2}[f(a)+f(b)] = T(f)}$ ... ③

//-> Error in trapezoidal rule:-

$$E = \frac{1}{2!} f''(\xi) \int_a^b (x-x_0)(x-x_1)\,dx, \quad \xi \in (x_0, x_1)$$

$$\Rightarrow E = \frac{1}{2} f''(\xi) \int_a^b (x-a)(x-b)\,dx$$

After solving, we get:

$$\boxed{E(f) = -\frac{1}{12}(b-a)^3 f''(\xi)} \quad a \le \xi \le b \quad \text{... ④}$$

//-> (n=2) Simpson's (1/3) rule:-

Here, no. of node is 3 i.e. x_0, x_1, x_2

$\therefore x_0 = a, \; x_1 = \frac{a+b}{2}, \; x_2 = b$

So, $I(P) = R_3(f) = w_0 f(a) + w_1 f\left(\frac{a+b}{2}\right) + w_2 f(b)$

$$\therefore w_0 = \int_a^b l_0(x)\,dx = \int_a^b \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}\,dx$$

Putting $x = x_0 + \alpha h$ & $x_i = x_0 + ih$ $(i=1,2)$, we've:

$$w_0 = \int_0^2 \frac{(\alpha-1)h(\alpha-2)h}{(-h)(-2h)}\,h\,d\alpha \Rightarrow \boxed{w_0 = \frac{h}{3}}$$

$$\therefore w_1 = \int_a^b l_1(x)\,dx = \int_a^b \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}\,dx$$

$$\Rightarrow \boxed{w_1 = \frac{4h}{3}}$$

$$\therefore w_2 = \int_a^b l_2(x)\,dx = \int_a^b \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1} dx$$

$$\Rightarrow \boxed{w_2 = \frac{h}{3}}$$

$$\therefore I(P) = \frac{h}{3} f(a) + \frac{4h}{3} f\left(\frac{a+b}{2}\right) + \frac{h}{3} f(b)$$

$$\Rightarrow I(P) = \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$= \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

So, $$\boxed{I(P) = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]} \quad \text{---} ⑤$$

This eqn. is called simpson's $\frac{1}{3}$ rule.
Also called "parabolic rule".

// -> Th^m :-

For a quadrature rule based on (n+1) nodes, the number (n) is defined to be the index of the rule.

ⓐ If (n) is even, & $f \in C^{n+2}[a,b]$, then:

$$E(f) = \frac{h^{n+3} f^{(n+2)}(\eta_1)}{(n+2)!} \int_0^n \mu^2(\mu-1)\cdots(\mu-n)\,d\mu$$

ⓑ If (n) is odd, & $f \in C^{n+1}[a,b]$, then:

$$E(f) = \frac{h^{n+2} f^{(n+1)}(\eta_2)}{(n+1)!} \int_0^n \mu(\mu-1)\cdots(\mu-n)\,d\mu$$

where, $\eta_1, \eta_2 \in [a,b]$ & $\boxed{h = \frac{b-a}{n}}$

//-> Error in Simpson's $\frac{1}{3}$ rule :-

As here, the index of simpson's $\frac{1}{3}$ rule is 2 (even), so, the error is given by:

$$E(f) = \frac{h^5}{4!} f^{iv}(\eta) \int_0^2 \mu^2(\mu-1)\cdot\cdot(\mu-2)\,d\mu$$

where, $h = \frac{b-a}{2}$

But, $\int_0^2 \mu^2(\mu-1)(\mu-2)\,d\mu = -\frac{4}{15}$

So, $E(f) = \frac{h^5}{24} f^{iv}(\eta) \cdot \left(\frac{-4}{15}\right) = \boxed{-\frac{h^5}{90} f^{iv}(\eta)}$

$$\Rightarrow \boxed{E(f) = -\frac{(b-a)^5}{2880} f^{iv}(\eta)}, \quad a \le \eta \le b \quad \text{---} ⑥$$

//-> (n = 3) Simpson's (3/8) rule :-

Here, no. of node is 4 i.e. x_0, x_1, x_2, x_3

$\therefore x_0 = a,\ x_1 = x_0 + h,\ x_2 = x_0 + 2h,\ x_3 = b$

where, $\boxed{h = \frac{b-a}{3}}$

So, $I(P) = R_4(f) = w_0 f_0 + w_1 f_1 + w_2 f_2 + w_3 f_3$

$$\therefore w_0 = \int_a^b l_0(x)\,dx = \int_a^b \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\,dx$$

$$= \int_0^3 \frac{(\alpha-1)h(\alpha-2)h(\alpha-3)h}{(-h)(-2h)(-3h)}\,h\,d\alpha$$

$$\Rightarrow \boxed{w_0 = \frac{3h}{8}}$$

$$\therefore \boxed{w_1 = \frac{9h}{8}\ ;\ w_2 = \frac{9h}{8}\ ;\ w_3 = \frac{3h}{8}}$$

$\therefore I(p) = \frac{3h}{8} f(x_0) + \frac{9h}{8} f(x_1) + \frac{9h}{8} f(x_2) + \frac{3h}{8} f(x_3)$

$$\Rightarrow \boxed{I(P) = \frac{3h}{8}\left[f(x_0) + 3\{f(x_1) + f(x_2)\} + f(x_3)\right]}$$

This equn. is called simpson's $\frac{3}{8}$ rule ⑦

II-> Error in simpson's (3/8) rule:-

As here, the index of simpson's $\left(\frac{3}{8}\right)$ rule is 3(odd), so, the error is given by:

$$E(f) = \frac{h^5}{4!} f^{iv}(\eta) \int_0^3 \mu(\mu-1)(\mu-2)(\mu-3)\, d\mu$$

where, $h = \frac{b-a}{3}$

But, $\int_0^3 \mu(\mu-1)(\mu-2)(\mu-3)\, d\mu = -\frac{9}{10}$

so, $E(f) = \frac{h^5}{24} f^{iv}(\eta) \cdot \left(-\frac{9}{10}\right) = -\frac{3h^5}{80} f^{iv}(\eta)$

$$\Rightarrow \boxed{E(f) = \frac{(b-a)^5}{6480} f^{iv}(\eta)}, \quad a \le \eta \le b$$

N.B: Relatⁿ between $S_{1/3}(f)$, $M(f)$, $T(f)$:-

$$\boxed{S_{1/3}(f) = \frac{2 \cdot M(f) + T(f)}{3}}$$

$S_{1/3}(f)$ is related to $M(f)$ & $T(f)$ in the ratio of 2 : 1 respectively.

Q: Find $\int_0^1 e^{-x^2} dx$ by Trapezoidal, $S_{1/3}(f)$ & $S_{3/8}(f)$.

Ans - $T(f) = \frac{1-0}{2}\left[e^{-0^2} + e^{-1^2}\right]$

$= 0.68393972059$

$S_{1/3}(f) = \frac{1-0}{6}\left[e^{-0^2} + 4e^{-(1/2)^2} + e^{-1^2}\right]$

$= 0.74718042891$

$$S_{3/8}(f) = \frac{1-0}{8}\left[e^{-0^2} + 3\left\{e^{-(1/3)^2} + e^{-(2/3)^2}\right\} + e^{-1^2}\right]$$

$= 0.74699231961$

II-> Degree of precis^n :-

Let $R_{n+1}(f)$ be an $(n+1)$ pt. rule, given by

$$\boxed{R_{n+1}(f) = w_0 f_0 + w_1 f_1 + \cdots + w_n f_n} \cdots ①$$

$$\Rightarrow I(f) = I(P) + E(f) \cdots ②$$

-> ① The rule $R_{n+1}(f)$ is said to be exact if $E(f) = 0$ in equn. ②. That means for the exactness, we consider the monomial (x^m) (where m is non-ve integer) such that $I(x^m) = R_{n+1}(x^m) + E(x^m)$, where $E(x^m) = 0 \Rightarrow I(x^m) = R_{n+1}(x^m)$.

-> ② The degree of precis^n of the rule $R_{n+1}(f)$ is denoted as (d) (+ve integer) if the rule is exact for all monomials of degree $\le d$, but fails to be exact for monomial of degree $(d+1)$; i.e. (d) is the degree of precis^n of rule $R_{n+1}(f)$ if:

$$\boxed{E(x^m) = \begin{cases} 0, & 0 \le m \le d \\ \text{non-zero}, & m = d+1 \end{cases}}$$

~~The error term will be exact if we takes f(x)~~

e.g. In Trapezoidal rule; $I(f) = R_2(f) + E(f) = R_2(f) - \frac{1}{12}(b-a)^3 f''(\eta)$

The error term will be zero if we take $f(x) = 1$ or x (i.e. $m = 0, 1$). Since $f''(1) = 0$ & $f''(x) = 0$. But for $f(x) = x^2$ $f''(x^2) \neq 0$. So, the degree of precis^n is 1 for $T(f)$.

The degree of precis^n of both simpson's 1/3 & 3/8 rule is 3, since their error terms involve 4th derivative of the integrand.

// → Compound quadrature rule (composite rule) :-

For large interval of integrat^n, we require compound quadrature rule.

// → Compound rectangular rule :-

Let the pts. are $x_0 = a, x_1, x_2, x_3, --- x_n = b$.
i.e. $[a, b]$ divided into (n) sub-equal parts or subintervals of equal length (h).

x_0
x_1
x_2
x_n
a
b

$$\left[h = \frac{b-a}{n} \right]$$

where, $x_i = x_0 + ih$, $i = 0(1)n$

∴ We know $I(f) = \int_a^b f(x)\,dx$. So, let's consider the ith subinterval $[x_{i-1}, x_i]$.

So, the integrat^n in this region is =

$$\int_{x_{i-1}}^{x_i} f(x)\,dx \simeq (x_i - x_{i-1}) f(x_{i-1})$$

[Using R'(f) rule]

$$\text{So, } I(f) = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\,dx$$

$$\Rightarrow I(f) = \sum_{i=1}^{n} \left[(x_i - x_{i-1}) f(x_{i-1}) + \frac{1}{2}(x_i - x_{i-1})^2 f'(\eta_i) \right]$$

where, $\eta_i \in (x_{i-1}, x_i)$

Leaving the error term, we've:

$$I_{CLR}(f) \simeq \sum_{i=1}^{n} \left[(x_i - x_{i-1}) f(x_{i-1}) \right]$$

$$\simeq (x_1 - x_0) f(x_0) + (x_2 - x_1) f(x_1) + \cdots + (x_n - x_{n-1}) f(x_{n-1})$$

$$\therefore \boxed{I_{CLR}(f) \simeq h\left[f(x_0) + f(x_1) + \cdots + f(x_{n-1})\right]}$$

The above eqn. is called compound left rectangular rule; & the error term is given by:

$$E_{CLR}(f) = \sum_{i=1}^{n} \frac{1}{2} (x_i - x_{i-1})^2 f'(\xi_i)$$

$$= \frac{1}{2}\left[(x_1 - x_0)^2 f'(\xi_1) + (x_2 - x_1)^2 f'(\xi_2) + \cdots + (x_n - x_{n-1})^2 f'(\xi_n)\right]$$

$$= \frac{h^2}{2}\left[f'(\xi_1) + f'(\xi_2) + \cdots + f'(\xi_n)\right]$$

$$\therefore \boxed{E_{CLR}(f) = \frac{nh^2}{2} f'(\mu)} \quad \text{---} \ (11)$$

where, $\mu \simeq \xi_1, \xi_2, \ldots, \xi_n$, $a \le \mu \le b$

$$\Rightarrow E_{CLR}(f) = \frac{n(b-a)^2}{2n^2} f'(\mu) \quad \left[\because h = \frac{b-a}{n}\right]$$

$$\Rightarrow \boxed{E_{CLR}(f) = \frac{(b-a)^2}{2n} f'(\mu)} \quad \text{---} \ (12)$$

Q: Evaluate $\int_0^1 x\,dx$ by using CLR rule, choosing $n = 2, 3, 4, 5, 6$

Ans- The given interval is $[0, 1]$ which let be ~~sub~~divided into (n) equal subintervals.

Here, $f(x) = x$ & $h = \frac{1-0}{n} = \frac{1}{n}$

$$\therefore R_{CLR}(f) = h\left[f(x_0) + f(x_1) + \cdots + f(x_{n-1})\right]$$

$$= \frac{1}{n}\left[0 + \frac{1}{n} + \frac{2}{n} + \cdots + \frac{n-1}{n}\right]$$

$$= \frac{1}{n^2}\left[1+2+\cdots+(n-1)\right]$$

$$= \frac{1}{n^2}\,\frac{n(n-1)}{2} \qquad \left[\because \sum_{n=1}^{k}(n) = \frac{n(n+1)}{2}\right]$$

$$= \frac{1}{2}\left[\frac{n-1}{n}\right] = \frac{1}{2}\left[1-\frac{1}{n}\right]$$

So,

n	$I(f) = \frac{1}{2}\left[1-\frac{1}{n}\right]$
2	$\frac{1}{2}\left[1-\frac{1}{2}\right] = 0.25$
3	$\frac{1}{2}\left[1-\frac{1}{3}\right] = 0.\overline{3}$
4	$\frac{1}{2}\left[1-\frac{1}{4}\right] = 0.375$
5	$\frac{1}{2}\left[1-\frac{1}{5}\right] = 0.4$
6	$\frac{1}{2}\left[1-\frac{1}{6}\right] = \frac{5}{12} = 0.41666$

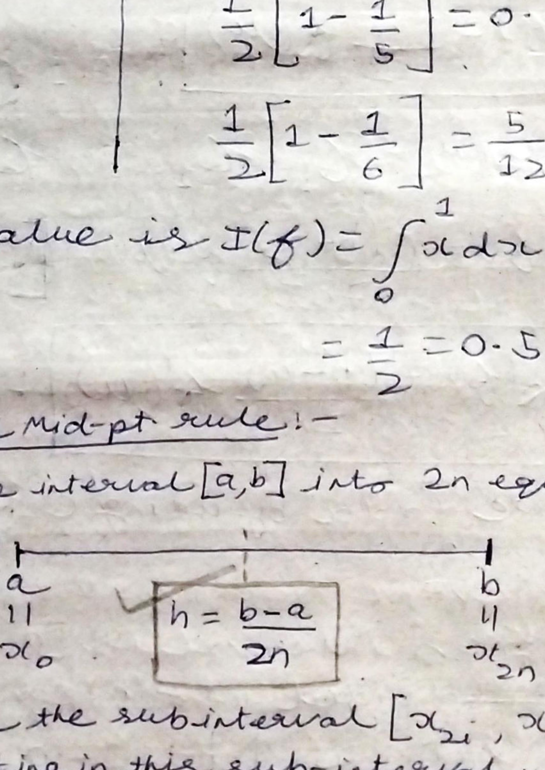
Exact value is $I(f) = \int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1$

$$= \frac{1}{2} = 0.5$$

II → Compound Mid-pt rule :-

Divide the interval $[a,b]$ into $2n$ equal parts

Interval = ~~Odd~~ Even
Node = Odd



$$\boxed{h = \frac{b-a}{2n}}$$

n = 1
2ep

consider the sub-interval $[x_{2i}, x_{2i+2}]$. Now, Integrating in this sub-interval, we've:

$\int_{x_{2i}}^{x_{2i+2}} f(x)\,dx$. Using mid-pt rule, we've:

$= (x_{2i+2} - x_{2i}) f(x_{2i+1})$ with error of

$$\frac{1}{4!}(x_{2i+2} - x_{2i})^3 f''(\xi_i), \quad \xi_i \in (x_{2i}, x_{2i+2})$$

So, $I_{CMR} = \sum_{i=0}^{n-1} \Big[(x_{2i+2} - x_{2i}) f(x_{2i+1}) + \frac{1}{4!}(x_{2i+2} - x$

$\cdot f''(\xi_i) \Big]$

Leaving the error term, we get:

$$I_{CMR} \simeq \sum_{i=0}^{n-1} (x_{2i+2} - x_{2i}) f(x_{2i+1})$$

$$= (x_2 - x_0) f(x_1) + (x_4 - x_2) f(x_3) + \cdots + (x_{2n} - x_{2n-2}) f(x_{2n-1})$$

$$= 2h\, f(x_1) + 2h\, f(x_3) + 2h\, f(x_5) + \cdots + 2h\, f(x_{2n-1}$$

$$\Rightarrow \boxed{I_{CMR}(f) \simeq 2h[f(x_1) + f(x_3) + \cdots + f(x_{2n-1})]}$$

This is called compound mid-pt rule & the error term is given by:

$$E_{CMR}(f) = \frac{1}{4!} \sum_{i=0}^{n-1} (x_{2i+2} - x_{2i})^3 f''(\xi_i)$$

$$= \frac{1}{24} \cdot (2h)^3 \sum_{i=0}^{n-1} f''(\xi_i)$$

$$= \frac{8h^3}{24} \cdot n f''(\mu), \text{ where } \mu \in [a, b]$$

$$\Rightarrow E_{CMR}(f) = \frac{h^3}{3} n f''(\mu) = \frac{(b-a)^3}{8n^3 \times 3} n f''(\mu)$$

$$\left[\because h = \frac{b-a}{2n}\right]$$

$$\Rightarrow \boxed{E_{CMR}(f) = \frac{(b-a)^3}{24n^2} f''(\mu)}$$ ⑮

Q: Evaluate $I(f) = \int_0^1 x^2\, dx$ by CMR taking $2n = 2, 4, 6, 8, 10$.

Ans - $M_{CMR}(f) = 2h[f(x_1) + f(x_3) + \cdots + f(x_{2n-1})]$

But, $h = \frac{b-a}{2n} = \frac{1-0}{2n} = \frac{1}{2n}$

$$\therefore \frac{1}{n}\left[\left(\frac{1}{2n}\right)^2 + \left(\frac{3}{2n}\right)^2 + \cdots + \left(\frac{2n-1}{2n}\right)^2\right]$$

$$= \frac{1}{4n^3}[1^2 + 3^2 + \cdots + (2n-1)^2]$$

$$= \frac{1}{4n^3} \cdot \frac{n}{3} \cdot (4n^2 - 1) = \frac{4n^2 - 1}{12n^2}$$

$\therefore$ $2n = 2 \Rightarrow n = 1 = 0.25$

$2n = 4 \Rightarrow n = 2 = 0.3125$

$2n = 6 \Rightarrow n = 3 = 0.32407$

$2n = 8 \Rightarrow n = 4 = 0.328125$

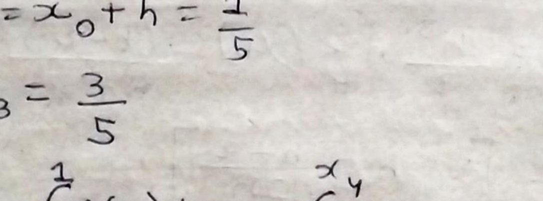
$2n = 10 \Rightarrow n = 5 = 0.\overline{3}$

& Exact $= 0.\overline{3}$

$$\boxed{\text{N.B: } R_{CRR}(f) = h[f(x_1) + f(x_2) + \cdots + f(x_n)]}$$

Q: Find $I(f) = \int_0^1 (x^2+1)\,dx$ by CMR dividing $[0, 1]$ into 5 sub-intervals.

Ans -



$\therefore h = \frac{1-0}{5} = \frac{1}{5}$

$\because x_0 = 0$

$\therefore x_1 = x_0 + h = \frac{1}{5}$

& $x_3 = \frac{3}{5}$

$$\therefore I(f) = \int_0^1 f(x)\,dx = \int_0^{x_4} f(x)\,dx + \int_{x_4}^1 f(x)\,dx$$

$$= 2h[f(x_1) + f(x_3)] + [(1 - x_4) f(x_4)]$$

$$= \frac{2}{5}\left[\frac{1}{25} + 1 + \frac{9}{25} + 1\right] + \left[\left(1 - \frac{4}{5}\right) \cdot \frac{41}{25}\right] = 1.288$$

Exact: $\int_0^1 (x^2+1)\,dx = 1.\overline{3}$

II-> Compound trapezoidal rule :-

Let the interval $[a, b]$ be divided into (n) subintervals i.e. $x_0 = a, x_1, x_2, \dots, x_n = b$.

where, $x_i = x_0 + ih$, $i = 0(1)n$ or $i \in [0, n]$

Let's consider the interval $[x_{i-1}, x_i]$.
So, integrating in this subinterval, we've

$$\int_{x_{i-1}}^{x_i} f(x)\,dx = \frac{(x_i - x_{i-1})}{2}\left[f(x_{i-1}) + f(x_i)\right]$$

(using trapezoidal rule)

with error : $-\frac{(x_i - x_{i-1})^3}{12} f''(\xi_i)$

$$x_{i-1} \le \xi_i \le x_i$$

So, $I_{CTR}(f) \simeq \sum_{i=1}^{n} \frac{(x_i - x_{i-1})}{2}\left[f(x_{i-1}) + f(x_i)\right]$

$$= \frac{1}{2}\Big[(x_1 - x_0)(f(x_0) + f(x_1)) + (x_2 - x_1)(f(x_1) + f(x_2)) + \dots + (x_n - x_{n-1})(f(x_{n-1}) + f(x_n))\Big]$$

$$\Rightarrow \boxed{I_{CTR}(f) = \frac{h}{2}\Big[f(x_0) + 2\{f(x_1) + f(x_2) + \dots + f(x_{n-1})\} + f(x_n)\Big]}$$

∴ This is called compound trapezoidal rule & the error term is given by :

$$E_{CTR}(f) = -\sum_{i=1}^{n} \frac{1}{12}(x_i - x_{i-1})^3 f''(\xi_i)$$

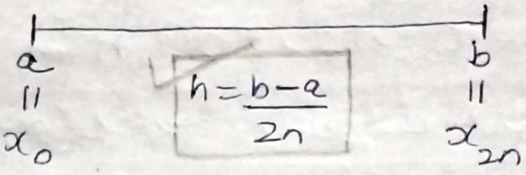
$$= -\frac{h^3}{12} n f''(\mu) \text{, where, } \mu \in (a, b)$$

$$= -\frac{(b-a)^3}{12n^3} n f''(\mu)$$

$$\therefore \boxed{E_{CTR}(f) = \frac{-(b-a)^3}{12n^2} f''(\mu)}$$

II-> compound Simpson's 1/3 rule:-

Divide the interval ~~into~~ $[a,b]$ into $2n$ equal parts.



$$\boxed{h = \frac{b-a}{2n}}$$

Consider the subinterval $[x_{2i}, x_{2i+2}]$. Now, integrating in this subinterval, we've:

$\int_{x_{2i}}^{x_{2i+2}} f(x)\,dx$. Using Simpson's $\frac{1}{3}$ rule, we've

$$= \frac{h}{3}\left[f(x_{2i}) + 4f(x_{2i+1}) + f(x_{2i+2})\right]$$

$$\text{So, } I_{CS1/3} = \int_a^b f(x)\,dx$$

$$= \sum_{i=0}^{n-1} \int_{x_{2i}}^{x_{2i+2}} f(x)\,dx$$

$$= \sum_{i=0}^{n-1} \frac{h}{3}\left[f(x_{2i}) + 4f(x_{2i+1}) + f(x_{2i+2})\right]$$

$$= \frac{h}{3}\Big[f(x_0) + f(x_2) + \dots + f(x_{2n-2}) + 4\{f(x_1) + f(x_3) + \dots + f(x_{2n-1})\} + f(x_2) + f(x_4) + \dots + f(x_{2n})\Big]$$

$$\therefore \boxed{I_{CS1/3}(f) = \frac{h}{3}\Big[f(x_0) + 2\{f(x_2) + f(x_4) + \dots + f(x_{2n-2})\} + 4\{f(x_1) + f(x_3) + \dots + f(x_{2n-1})\} + f(x_{2n})\Big\}}$$

This is known as compound Simpson's $\frac{1}{3}$

rule: And the error term is given by

$$E_{CS1/3}(f) = -\frac{h^5}{90}\sum_{i=0}^{n-1} f^{iv}(x_i)$$

$$= \frac{-h^5}{90}\left[n f^{iv}(\mu)\right], \text{ where, } a<\mu<b$$

$$\Rightarrow E_{CS1/3}(f) = -\frac{(b-a)^5 \cdot n \cdot f^{iv}(\mu)}{90\times(2n)^5}$$

$$\Rightarrow \boxed{E_{CS1/3}(f) = -\frac{(b-a)^5 f^{iv}(\mu)}{(2^5\times 90)n^4}}$$

Q: Solve $\int_0^1 \frac{dx}{1+x}$ by $T_{CTR}(f)$ taking $h=0.125$ or $\underline{n=8}$

Ans- $x_0 = 0$

$x_1 = h = 0.125$

$x_2 = 2h = 0.25$

$x_3 = 3h = 0.375$

$x_4 = 4h = 0.5$

$x_5 = 5h = 0.625$

$x_6 = 6h = 0.75$

$x_7 = 7h = 0.875$

$x_8 = 8h = 1$

$\therefore T_{CTR}(f) = 0.694122$

& Exact: 0.69314718056

Q: Solve $\int_0^6 \frac{dx}{1+x^2}$ by $T_{CTR}(f)$ taking $h=1$ or $\underline{n=6}$

Ans- $x_0 = 0$

$x_1 = h = 1$

$x_2 = 2h = 2$

$x_3 = 3h = 3$

$x_4 = 4h = 4$

$x_5 = 5h = 5$

$x_6 = 6h = 6$

$\therefore T_{CTR}(f) = 1.4107985$

& Exact: 1.40564764938

Q: Solve $\int_0^1 \frac{dx}{1+x^2}$ by $I_{CS1/3}(f)$ taking $h=\frac{1}{6}$ or $n=6$

Ans:- $x_0 = 0$

1 $x_1 = h = 0.167$

2 $x_2 = 2h = 0.33$ $\therefore I_{CS1/3}(f) = 0.7857679$

3 $x_3 = 3h = 0.5$ & Exact : 0.7853981

4 $x_4 = 4h = 0.67$

5 $x_5 = 5h = 0.83$

6 $x_6 = 1$

Q: Solve $\int_0^4 e^x dx$ taking $h = 1$ or $n = 4$

Ans:- $x_0 = 0$

1 $x_1 = h = 1$ $\therefore I_{CS1/3}(f) = 53.863845$

2 $x_2 = 2h = 2$ & Exact : 53.59815

3 $x_3 = 3h = 3$

4 $x_4 = 4$

II → Gauss-quadrature rule:-

We know the rule for the numerical evaluatⁿ of the integral $I(f) = \int_a^b f(x)dx$ is $R_{n+1}(f) = w_0 f_0 + w_1 f_1 + \cdots + w_n f_n$

ⓐ This Newton-cotes quadrature rule involves $2n+2$ parameters, out of which $w_0, w_1, \ldots, w_n$ are the $(n+1)$ coefficients & $x_0, x_1, \ldots, x_n$ are the $(n+1)$ nodes. But, we fix $(n+1)$ nodes in advance while calculating $R_{n+1}(f)$ only to calculate $w_0, w_1, \ldots, w_n$

ⓐ But, in case of gaussian quadrature rule, we treat all the $(2n+2)$ parameters as unknowns & try to determine them such that the rule has degree of precisⁿ as high as possible.

So, an $(n+1)$ gauss rule has degree of precisⁿ as $2n+1$.

$$\therefore I(f) = \int_a^b f(x)dx = \int_a^b w(x) g(x) dx \quad ①$$

where, $w(x) = \frac{f(x)}{g(x)}$ is called wt. funct^n

which is non-ve & integrable in $[a, b]$.

There are 4 types of gauss rule. They are:

① **Gauss-Legendre rule**

If $w(x) = 1$ & range of integrat^n is $[-1, 1]$.

② **Gauss-Laguerre rule**

If $w(x) = e^{-x}$ & range $= [0, \infty)$

③ **Gauss-Hermite rule**

If $w(x) = e^{-x^2}$, & range $= (-\infty, \infty)$

④ **Gauss-Chebyshev rule**

If $w(x) = (1-x^2)^{-1/2}$ & range $= [-1, 1]$

// → **Gauss-Legendre rule:-**

If the range isn't $[-1, 1]$ in an integral with finite range $[a, b]$, we can transform it to be so by a suitable linear transform & apply gauss-legendre rule.

Let's take $\boxed{t = \frac{[2x - (a+b)]}{b-a}}$

Then the interval $[a, b]$ transformed onto $[-1; 1]$. So, $I(f) = \int_{-1}^{1} f(x)\,dx$

$= \int_{-1}^{1} g(x)\,dx$ $(\because w(x) = 1)$ or the wt. funct^n $w(x) = 1$

// → **2-pt Gauss-Legendre rule:-**

$R_2(f) = w_0 f(x_0) + w_1 f(x_1)$ - - - ①

Now, $I(f) = R_2(f) + E_2(f)$, where $E_2(f) =$ error

Here, there are 4 unknowns w_0, w_1, x_0, x_1.

So to find the unknowns, we've to make eqn ① exact.

$\therefore E_2(x^i)=0$, $i=0,1,2,3$ $(1, x, x^2, x^3$ monomials$)$

i.e. $E_2(1)=0$, $E_2(x)=0$, $E_2(x^2)=0$, $E_2(x^3)=0$

As a result, we've:

$$\left.\begin{aligned} w_0+w_1&=2 \\ w_0x_0+w_1x_1&=0 \\ w_0x_0^2+w_1x_1^2&=2/3 \\ w_0x_0^3+w_1x_1^3&=0 \end{aligned}\right\} ⓘⓘ$$

Solving, we get: $w_0=w_1=1$ & $x_0=\frac{1}{\sqrt{3}}$, $x_1=-\frac{1}{\sqrt{3}}$. So, 2-pt. gauss-legendre rule is given by:

$$\boxed{R_2(f)=f\left(-\frac{1}{\sqrt{3}}\right)+f\left(\frac{1}{\sqrt{3}}\right)}$$

The degree of precisⁿ of this rule is 3.

// → 3-pt Gauss-Legendre rule:-

$$R_3(f)=w_0f(x_0)+w_1f(x_1)+w_2f(x_2) \cdots ①$$

Now, $I(f)=R_3(f)+E_3(f)$. Here, there are 6 unknowns $w_0, w_1, w_2, x_0, x_1, x_2$. So to find the unknown, we've to make eqn ① exact.

$\therefore E_3(x^i)=0$, $i=0,1,2,3,4,5$ $(1, x, x^2, x^3, x^4, x^5$

$\Rightarrow E_3(1)=E_3(x)=E_3(x^2)\pm E_3(x^3)=E_3(x^4)$
$=E_3(x^5)=0$.

As a result, we've:

$$w_0+w_1+w_2=2$$
$$w_0x_0+w_1x_1+w_2x_2=0$$
$$w_0x_0^2+w_1x_1^2+w_2x_2^2=2/3$$

$w_0 x_0^3 + w_1 x_1^3 + w_2 x_2^3 = 0$

$w_0 x_0^4 + w_1 x_1^4 + w_2 x_2^4 = 2/5$

$w_0 x_0^5 + w_1 x_1^5 + w_2 x_2^5 = 0$

So, $w_0 = \frac{8}{9}$, $w_1 = w_2 = \frac{5}{9}$, $x_0 = 0$, $x_1 = -x_2 = \sqrt{\frac{3}{5}}$

So, 3 pt. gauss-legendre rule is given by

$$\boxed{R_3(f) = \frac{1}{9}\left[8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right) + 5f\left(-\sqrt{\frac{3}{5}}\right)\right]}$$

The degree of precisn of this rule is 5.

Q: Evaluate $I(f) = \int_0^1 x e^{-x} dx$ approximately.

Ans- $t = \frac{2x-1}{1} = 1 \Rightarrow x = \frac{t+1}{2} \Rightarrow dx = \frac{dt}{2}$

$$\therefore \int_{-1}^{1} \frac{t+1}{2} e^{-\left(\frac{t+1}{2}\right)} \frac{dt}{2} = \frac{1}{4} \int_{-1}^{1} (t+1) e^{-\left(\frac{t+1}{2}\right)} dt$$

$$= \int_{-1}^{1} g(t)\, dt$$

$\therefore R_2(f) = 0.264740022422$

$\therefore R_3(f) = 0.2642424278969$

11-> Numerical solutn of Differential equn:-

- The linear differential equn. of 1st order is $\frac{dy}{dx} = f(x,y)$, $x_0 \le x \le b$, $y(x_0) = y_0$ - ①
- Equn ① is known as the initial value problem.
- There are some methods to solve this equn below.

11-> Picard's method:-

$$\frac{dy}{dx} = f(x,y) \Rightarrow dy = f(x,y)\,dx$$

$$\Rightarrow \int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\,dx$$

$$\Rightarrow y = y_0 + \int_{x_0}^{x} f(x,y)\,dx$$

Let, $y = y_1$

$$\Rightarrow y_1 = y_0 + \int_{x_0}^{x} f(x,y_0)\,dx \quad \text{(1st approx.)}$$

Similarly, $y_2 = y_0 + \int_{x_0}^{x} f(x,y_1)\,dx$ (2nd approx.)

Hence, $\boxed{y_{n+1} = y_0 + \int_{x_0}^{x} f(x,y_n)\,dx}$

Q: Solve $\frac{dy}{dx} = x+y$, $y(0)=1$ by Picard's method.

Ans- $x_0 = 0$, $y_0 = 1$

$$\therefore y_1 = y_0 + \int_{x_0}^{x} f(x,y_0)\,dx$$

$$= 1 + \int_{0}^{x} (x+1)\,dx = 1 + x + \frac{x^2}{2}$$

$$\therefore y_2 = y_0 + \int_{x_0}^{x} f(x,y_1)\,dx$$

$$= 1 + \int_{0}^{x} \left[x + 1 + x + \frac{x^2}{2}\right] dx$$

$$= 1 + x + x^2 + \frac{x^3}{6}$$

$$\therefore y_3 = y_0 + \int_{x_0}^{x} f(x,y_2)\,dx$$

$$= 1 + \int_{0}^{x} \left[1 + x + x^2 + \frac{x^3}{6} + x\right] dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

Exact solut^n is:

$$\frac{dy}{dx} = x + y \Rightarrow \frac{dy}{dx} - y = x$$

$$\therefore I.F = e^{\int -dx} = e^{-x}$$

$$\Rightarrow ye^{-x} = \int x e^{-x} + c$$

$$= -xe^{-x} - e^{-x} + c$$

$$\therefore y = -x - 1 + ce^{x}$$

when $x = 0, y = 1$

$$\therefore 1 = 0 - 1 + ce^{0} \Rightarrow c = 2$$

$$\therefore y = -x - 1 + 2e^{-x} = 2e^{-x} - x - 1$$

II-> Euler's Method:-

Expanding $y(x)$ in taylor series, we've:

$$y(x_0 + h) = y(x_0) + hy'(x_0) + \frac{h^2}{2} y''(\xi_0)$$

$$= y(x_0) + (x_1 - x_0) y'(x_0) + \frac{(x_1 - x_0)^2}{2} y''(\xi_0)$$

where, $x_0 < \xi < x_1$

Now, neglecting the remainder term, we've:

$$y(x_1) = y(x_0) + hy'(x_0)$$

$$\Rightarrow y_1 = y_0 + hf(x_0, y_0) \quad \left[\because \frac{dy}{dx} = y' = f(x, y)\right]$$

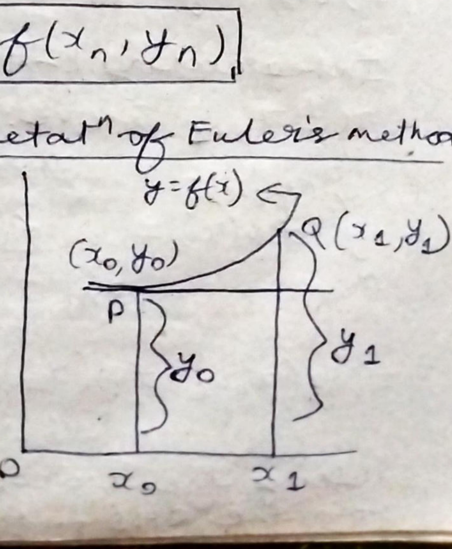
Similarly, $y_2 = y_1 + hf(x_1, y_1)$

In general $\boxed{y_{n+1} = y_n + hf(x_n, y_n)}$

II-> Geometrical interpretat^n of Euler's method:-

The equ^n. of the tangent line at (x_0, y_0) to the solut^n ~~at (x_0, y_0)~~ curve $y = f(x)$ is:

$$y - y_0 = \left.\frac{dy}{dx}\right]_{[x_0, y_0]} (x - x_0)$$



But, $\left.\frac{dy}{dx}\right|_{[x_0, y_0]} = f(x_0, y_0)$

$\therefore y - y_0 = f(x_0, y_0)(x - x_0)$

But y & x coordinates of pt. Q is y_1 & x_1 respectively.

$\therefore y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$

$\Rightarrow y_1 = y_0 + h\,f(x_0, y_0)$

similarly, $y_2 = y_1 + h\,f(x_1, y_1)$

Q: Solve $\frac{dy}{dx} = x + y$, $y(0) = 1$ by euler's method, taking $h = 0.1$.

Ans - $y_1 = y_0 + h\,f(x_0, y_0)$

$= 1 + 0.1(0 + 1) = 1.1$

$y_2 = y_1 + h\,f(x_1, y_1)$

$= 1.1 + 0.1(0.1 + 1.1) = 1.22$

$y_3 = y_2 + h\,f(x_2, y_2)$

$= 1.22 + 0.1(0.2 + 1.22) = 1.362$

II -> **Modified Euler's Method:-**

$\frac{dy}{dx} = f(x, y)$ is the given differential equatn.

$\Rightarrow dy = f(x, y)\,dx$

Integrating both sides from $x = x_n$ to $x = x_{n+1}$, we've:

$$\int_{x_n}^{x_{n+1}} dy(x) = \int_{x_n}^{x_{n+1}} f(x, y)\,dx$$

$$\Rightarrow y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(x, y)\,dx$$

Using the trapezoidal rule upon the integral on the RHS of the above eqn, we've:

$y(x_{n+1}) = y(x_n) + \frac{h}{2}[f(x_n, y(x_n)) + f(x_{n+1}, y(x_{n+1}))]$

$\Rightarrow y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$

where, y_n & y_{n+1} are approximatⁿs to $y(x_n)$ & $y(x_{n+1})$ respectively.

The expressⁿ on the RHS of this approximatⁿ is defined to be the approximatⁿ to $y(x_{n+1})$ by modified Euler's method.

If we use Y_n for denoting the approximatⁿs, the above can be written as:

$$\boxed{Y_{n+1} = Y_n + \frac{h}{2}\{f(x_n, Y_n) + f(x_{n+1}, y_{n+1})\}} \quad \text{---} ①$$

we slightly modified it in the following manner.

For, $n = 0$, let $Y_0 = y_0$

& then $y_1 = Y_0 + h f(x_0, y_0)$

$y_2 = Y_1 + h f(x_1, y_1)$ Y_n

similarly, $\boxed{y_{n+1} = Y_n + h f(x_n, y_n)}$ --- ②

Q: Solve numerically $\frac{dy}{dx} = y + 1$, $y(0) = 1$ by modified Euler's method for $0 \le x \le 0.5$ taking $h = 0.1$.

Ans- $Y_0 = y_0 = 1$ & $x_0 = 0$

(0.1) ⓐ $y_1 = Y_0 + h f(x_0, y_0) = 1.2$

$Y_1 = Y_0 + \frac{h}{2}\{f(x_0, Y_0) + f(x_1, y_1)\} = 1.21$

(0.2) ⓐ $y_2 = Y_1 + h f(x_1, y_1) = 1.43$

$Y_2 = Y_1 + \frac{h}{2}\{f(x_1, Y_1) + f(x_2, y_2)\} = 1.442$

(0.3) ⓐ $y_3 = Y_2 + h f(x_2, y_2) = 1.685$

$Y_3 = Y_2 + \frac{h}{2}\{f(x_2, Y_2) + f(x_3, y_3)\}$

$= 1.69835$

ⓐ $y_4 = Y_3 + hf(x_3, y_3)$

(0.4) $= 1.966685$

$Y_4 = Y_3 + \frac{h}{2}\{f(x_3, Y_3) + f(x_4, y_4)\}$

$= 1.98161$

ⓐ $y_5 = Y_4 + hf(x_4, y_4) = 2.278295$

(0.5)

$Y_5 = Y_4 + \frac{h}{2}\{f(x_4, Y_4) + f(x_5, y_5)\}$

$= 2.29460525$

& $y(0.1) = 1.21034183615$

$y(0.2) = 1.44280551632$

$y(0.3) = 1.69971761515$

$y(0.4) = 1.98364939528$

$y(0.5) = 2.29744254114$

Q: $\frac{dy}{dx} = x + y$, $y(0) = 1$, $h = 0.25$ by modified Euler's method.

Ans:- $Y_0 = y_0 = 1$

ⓐ $x_1 = h = 0.25$

$y_1 = Y_0 + hf(x_0, y_0) = 1.25$

$Y_1 = Y_0 + \frac{h}{2}\{f(x_0, Y_0) + f(x_1, Y_1)$

$= 1.3125$

ⓐ $x_2 = 2h = 0.5$

$y_2 = Y_1 + hf(x_1, y_1) = 1.6875$

$Y_2 = Y_1 + \frac{h}{2}\{f(x_1, Y_1) + f(x_2, y_2)\}$

$= 1.78125$

ⓐ $x_3 = 3h = 0.75$

$y_3 = Y_2 + hf(x_2, y_2) = 2.328125$

$Y_3 = Y_2 + \frac{h}{2}\{f(x_2, Y_2) + f(x_3, y_3)\}$

$= 2.451171875$

ⓐ $x_4 = 4h = 1$

$y_4 = Y_3 + h f(x_3, y_3) = 3.220703125$

$Y_4 = Y_3 + \frac{h}{2}\{f(x_3, Y_3) + f(x_4, y_4)\}$

$= 3.37890625$

//→ **Runge-Kutta Method:-**

The runge-kutta methods are widely used for solving the initial value problem given by eqn. $\frac{dy}{dx} = f(x,y)$

//→ **2nd order RK formula:-**

$y_{n+1} = y_n + aK_1 + bK_2$, where:

$K_1 = h f(x_n, y_n)$

$K_2 = h f(x_n + \alpha h, y_n + \beta K_1)$

& a, b, α, β are constants. Using Taylor's series & comparing the coefficients of h & h^2, we've:

$a + b = 1$

$b\alpha = b\beta = 1/2$

To find a, b, α, β, we've the following methods:

ⓐ **Improved tangent method**

$a = 0,\ b = 1;\ \alpha = \beta = 1/2$

ⓐ **Heun's method**

$a = b = 1/2;\ \alpha = \beta = 1$

ⓐ **Optimal method**

$a = \frac{1}{4};\ b = \frac{3}{4};\ \alpha = \beta = \frac{2}{3}$

/⇒ Using solutⁿ of improved tangent method

$$\boxed{y_{n+1} = y_n + h f\left(x_n + \frac{h}{2},\ y_n + \frac{h}{2} f(x_n, y_n)\right)}$$

[This eqn. is called 2pt. RK method using improved tangent method or midpt. method]

$$\boxed{y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_n + hf(x_n, y_n))\right]}$$

[This equn. is called Heun's formula, also called modified euler's method. (IMP)

I=> Using solutn of Optimal method

$$\boxed{y_{n+1} = y_n + \frac{h}{4}\left[f(x_n, y_n) + 3f\left(x_n + \frac{2h}{3}, y_n + \frac{2h}{3} f(x_n, y_n)\right)\right]}$$

Q: Solve the diff. equn. $\frac{dy}{dx} = y$, $y(0) = 1$ & $h = 0.1$ by 2 pt. RK method.

Ans- $y_0 = 1$, $x_0 = 0$

$\therefore y_1 = y_0 + hf\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right]$

$= 1.105 = y(0.1)$

$\therefore y_2 = y_1 + hf\left[x_1 + \frac{h}{2}, y_1 + \frac{h}{2} f(x_1, y_1)\right]$

$= 1.221025 = y(0.2)$

& Exact, $y(0.1) = 1.105$

$y(0.2) = 1.221$

So, error $\simeq 0$

II->RK method of order 3:-

In general, the 3rd order RK formula is given by:

$y_{n+1} = y_n + w_1 K_1 + w_2 K_2 + w_3 K_3$, where:

$K_1 = hf(x_n, y_n)$

$K_2 = hf(x_n + \alpha h, y_n + \beta K_1)$

$K_3 = hf(x_n + \gamma h, y_n + \delta_1 K_1 + \delta_2 K_2)$

The parameters $w_1, w_2, w_3, \alpha, \beta, \gamma, \delta_1, \delta_2$ have been found out by following formula:

(a) Heun's formula

$w_1 = \frac{1}{4}$, $w_2 = 0$, $w_3 = \frac{3}{4}$, $\alpha = \beta = \frac{1}{3}$, $\gamma = \frac{2}{3}$, $\delta_1 = 0$

$\delta_2 = \frac{2}{3}$

ⓑ Classical formula

$w_1 = \frac{1}{6}, w_2 = \frac{4}{6}, w_3 = \frac{1}{6}, \alpha = \beta = \frac{1}{2}, \gamma = 1,$

$\delta_1 = -1, \delta_2 = 2$

Hence, the 3 order RK methods are given by:

I=> Using Heun's formula

$$\boxed{y_{n+1} = y_n + \frac{1}{4}(K_1 + 3K_3)} \quad 3+1=4$$

where, $K_1 = hf(x_n, y_n)$

$K_2 = hf(x_n + \frac{h}{3}, y_n + \frac{K_1}{3})$

$K_3 = hf(x_n + \frac{2h}{3}, y_n + \frac{2K_2}{3})$

$n = 0, 1, 2, \ldots \quad (n \in W)$

II=> Using classical formula

$$\boxed{y_{n+1} = y_n + \frac{1}{6}(K_1 + 4K_2 + K_3)} \quad 1+4+1=6$$

where, $K_1 = hf(x_n y_n)$

$K_2 = hf(x_n + \frac{h}{2}, y_n + \frac{K_1}{2})$

$K_3 = hf(x_n + h, y_n - K_1 + 2K_2)$

$(n \in W)$

III-> RK method of order 4:-

The 4th order RK formula has the following general form.

$y_{n+1} = y_n + w_1 K_1 + w_2 K_2 + w_3 K_3 + w_4 K_4$

where, $K_1 = hf(x_n, y_n)$

$K_2 = hf(x_n + \alpha h, y_n + \beta K_1)$

$K_3 = hf(x_n + \gamma h, y_n + \delta_1 K_1 + \delta_2 K_2)$

$K_4 = hf(x_n + \mu h, y_n + \nu_1 K_1 + \nu_2 K_2 + \nu_3 K_3)$

where, the parameters $w_1, w_2, w_3, w_4, \alpha, \beta, \gamma,$

$\delta_1, \delta_2, \mu, \nu_1, \nu_2, \nu_3$ have been found out for diff. 4th order RK methods.

ⓐ Kutta's formula

$w_1 = w_4 = \frac{1}{8}, \quad w_2 = w_3 = \frac{3}{8}, \alpha = \beta = \frac{1}{3}, \gamma = \frac{2}{3}$

$\delta_1 = -\frac{1}{3}, \delta_2 = 1, \mu = 1, \nu_1 = \nu_3 = 1, \nu_2 = -1$

ⓐ classical formula

$w_1 = w_4 = \frac{1}{6}, \quad w_2 = w_3 = \frac{2}{6}, \alpha = \beta = \gamma = \frac{1}{3}, \delta_1 = 0,$

$\delta_2 = \frac{1}{2}, \mu = 1, \gamma_1 = \gamma_2 = 0, \gamma_3 = 1.$

Hence, the 4th order RK method is given by:

/=> Using Kutta's formula

$$\boxed{y_{n+1} = y_n + \frac{1}{8}(K_1 + 3K_2 + 3K_3 + K_4)} \quad 1+3+3+1=8$$

where, $K_1 = hf(x_n, y_n)$

$K_2 = hf\left(x_n + \frac{h}{3}, y_n + \frac{K_1}{3}\right)$

$K_3 = hf\left(x_n + \frac{2h}{3}, y_n - \frac{K_1}{3} + K_2\right)$

$K_4 = hf(x_n + h, y_n + K_1 - K_2 + K_3)$

$(n \in W)$

/=> Using classical formula

$$\boxed{y_{n+1} = y_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)} \quad 1+2+2+1 = 6$$

where, $K_1 = hf(x_n, y_n)$

$K_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{K_1}{2}\right)$

$K_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{K_2}{2}\right)$

$K_4 = hf(x_n + h, y_n + K_3) \quad (n \in W)$

Q: Solve $\frac{dy}{dx} = -2xy^2, y(0) = 1$ by 4th order RK method to evaluate $y(0.1)$ & $y(0.2)$

Ans- $y_0 = 1$; $x_0 = 0$; $h = 0.1$

ⓐ 1st approx

$\therefore y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$K_1 = hf(x_0, y_0) = 0$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = -0.01$

$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = -0.00990025$

$K_4 = hf(x_0 + h, y_0 + K_3) = -0.0196059503$

$\Rightarrow y_1 = 0.99019892495$

⊗ 2nd approx

$\therefore y_2 = y_1 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$K_1 = hf(x_1, y_1) = -0.0196\overset{0591762}{\cancel{7119385}}$

$K_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right) = -0.0288\overset{2940544}{\cancel{8075774}}$

$K_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right) = -0.028\overset{55879105}{\cancel{6094309}}$

$K_4 = hf(x_1 + h, y_1 + K_3) = -0.03\overset{698237716}{\cancel{7046999661}}$

$\Rightarrow y_2 = \cancel{0.962337782154}$
0.96153814366

VI → Numerical method for system of diff. equn:-

Let's consider a system of 2 ordinary diff. equns:

$$\frac{dy}{dx} = f_1(x, y, z) \quad ---- \text{①}$$

$$\frac{dz}{dx} = f_2(x, y, z) \quad ---- \text{②}$$

$y(x_0) = y_0$ & $z(x_0) = z_0$

Let (h) be the step length & $x_1 = x_0 + h$. Applying 2nd order RK formula due to Heun to the given equ^n, we've:

$K_1 = hf_1(x_n, y_n, z_n)$
$L_1 = hf_2(x_n, y_n, z_n)$
$K_2 = hf_1(x_n + h, y_n + K_1, z_n + L_1)$
$L_2 = hf_2(x_n + h, y_n + K_1, z_n + L_1)$

Then the 2nd order RK Heun's formula

Implies that:

$$\boxed{\begin{aligned} y_{n+1} &= y_n + \frac{1}{2}(K_1 + K_2) \\ z_{n+1} &= z_n + \frac{1}{2}(L_1 + L_2) \end{aligned}}$$

// → **By RK method of 3rd order :-**

Suppose the system of 2 ordinary diff. equn. be:

$$\frac{dy}{dx} = f_1(x, y, z)$$

$$\frac{dz}{dx} = f_2(x, y, z)$$

such that $y(x_0) = y_0$, & $z(x_0) = z_0$

By RK method of 3rd order (classical formula):

$$K_1 = h f_1(x_n, y_n, z_n)$$

$$L_1 = h f_2(x_n, y_n, z_n)$$

$$K_2 = h f_1\left(x_n + \frac{h}{2}, y_n + \frac{K_1}{2}, z_n + \frac{L_1}{2}\right)$$

$$L_2 = h f_2\left(x_n + \frac{h}{2}, y_n + \frac{K_1}{2}, z_n + \frac{L_1}{2}\right)$$

$$K_3 = h f_1(x_n + h, y_n - K_1 + 2K_2, z_n - L_1 + 2L_2)$$

$$L_3 = h f_2(x_n + h, y_n - K_1 + 2K_2, z_n - L_1 + 2L_2)$$

Thus, we've the solutn as:

$$\boxed{\begin{aligned} y_{n+1} &= y_n + \frac{1}{6}(K_1 + 4K_2 + K_3) \\ z_{n+1} &= z_n + \frac{1}{6}(L_1 + 4L_2 + L_3) \end{aligned}} \quad (n \in W)$$

// → **By RK method of 4th order :-**

Suppose the system of 2 ordinary diff equn. be:

$$\frac{dy}{dx} = f_1(x, y, z)$$

$$\frac{dz}{dx} = f_2(x, y, z)$$

such that $y(x_0) = y_0$ & $z(x_0) = z_0$

By RK method of 4th order (classical formula):

$$K_1 = h f_1(x_n, y_n, z_n)$$

$$L_1 = h f_2(x_n, y_n, z_n)$$

$$K_2 = h f_1\left(x_n + \frac{h}{2}, y_n + \frac{K_1}{2}, z_n + \frac{L_1}{2}\right)$$

$$L_2 = h f_2\left(x_n + \frac{h}{2}, y_n + \frac{K_1}{2}, z_n + \frac{L_1}{2}\right)$$

$$K_3 = h f_1\left(x_n + \frac{h}{2}, y_n + \frac{K_2}{2}, z_n + \frac{L_2}{2}\right)$$

$$L_3 = h f_2\left(x_n + \frac{h}{2}, y_n + \frac{K_2}{2}, z_n + \frac{L_2}{2}\right)$$

$$K_4 = h f_1(x_n + h, y_n + K_3, z_n + L_3)$$

$$L_4 = h f_2(x_n + h, y_n + K_3, z_n + L_3)$$

Thus, we've the solutⁿ as:

$$\boxed{\begin{aligned} y_{n+1} &= y_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) \\ z_{n+1} &= z_n + \frac{1}{6}(L_1 + 2L_2 + 2L_3 + L_4) \end{aligned}} \quad (n \in W)$$

~~Q: $\frac{dy}{dx} = y$; $y(0) = 1$, $h = 0.1$. Solve using improved-tangent method.~~

~~Ans: $y_0 = 1$ & $x_0 = 0$~~

~~$\therefore y_1 =$~~

Q: $\frac{dy}{dx} = x + y + z$ $\qquad y_0 = 1 = z_0$

$x_0 = 0$ & $h = 0.1$

$\frac{dz}{dx} = x + y - z$ $\qquad$ Solve by using 2nd order RK Heun's formula

$$\therefore K_1 = h f_1(x_0, y_0, z_0) = 0.2$$

$$L_1 = h f_2(x_0, y_0, z_0) = 0$$

$K_2 = hf_1(x_0+h, y_0+K_1, z_0+L_1) = 0.23$

$L_2 = hf_2(x_0+h, y_0+K_1, z_0+L_1) = 0.03$

$\therefore y_1 = y_0 + \frac{K_1}{2} + \frac{K_2}{2} = 1.215$

$z_1 = z_0 + \frac{L_1}{2} + \frac{L_2}{2} = 1.015$

Q: $\frac{dy}{dx} = -0.5y$ $\quad x_0 = 0 \quad z_0 = 6$

$\quad y_0 = 4$

$\frac{dz}{dx} = 4 - 0.3z - 0.1y \quad h = 0.5$

solve by using RK method of 4th order.

Ans- $K_1 = hf_1(x_0, y_0, z_0) = -1$

$L_1 = hf_2(x_0, y_0, z_0) = 0.9$

$K_2 = hf_1\left(x_0+\frac{h}{2}, y_0+\frac{K_1}{2}, z_0+\frac{L_1}{2}\right) = -0.875$

$L_2 = hf_2\left(x_0+\frac{h}{2}, y_0+\frac{K_1}{2}, z_0+\frac{L_1}{2}\right) = 0.8575$

$K_3 = hf_1\left(x_0+\frac{h}{2}, y_0+\frac{K_2}{2}, z_0+\frac{L_2}{2}\right) = -0.89$

$L_3 = hf_2\left(x_0+\frac{h}{2}, y_0+\frac{K_2}{2}, z_0+\frac{L_2}{2}\right) = 0.8575$
625

$K_4 = hf_1(x_0+h, y_0+K_3, z_0+L_3) = -0.7775$

$L_4 = hf_2(x_0+h, y_0+K_3, z_0+L_3) = 0.81586$

$\therefore y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$= 3.1154$

$\therefore z_1 = z_0 + \frac{1}{6}(L_1 + 2L_2 + 2L_3 + L_4)$

$= 6.85766$

(1→) **Methods for higher order diff. equn:**

Consider the differential equn:

$$\frac{d^2y}{dx^2} = f(x,y,y') \Rightarrow y'' = f(x,y,y')$$

Given, $y(x_0) = y_0$ & $y'(x_0) = y'_0$

Let's put $y' = z \Rightarrow y'' = z'$

$\therefore y' = z$ — — — — ①

& $y'' = f(x,y,y') \Rightarrow z' = f(x,y,z)$ .. ②

And, $y'(x_0) = y'_0 \Rightarrow z(x_0) = z_0$

Now, by solving equn. ① & ② by RK method of 2nd, 3rd & 4th order for system of diff. equns, we will get it's solutn.

II-> Multi step method:-

- Multi step methods are used for the numerical solutn of ordinary differential equatn.
- Single step methods such as Euler's method refer to only 1 previous pt. & its derivative to determine the current value.
- Methods such as RK-method take some intermediate steps to obtain a higher order method, but then discards all previous informatn before taking a 2nd step.
- Multistep methods attempt to gain efficiency by keeping & using the info from previous steps rather than discarding it. Consequently multistep methods refer to several previous pts. & derivative values.

We'll discuss 2 types of multistep methods for solving 1st order diff. equns

-> ⓐ Milnes Method

-> ⓑ Adam's ~~Bashforth~~ Method

II-> Milnes Method

Let's consider the ordinary diff. equn.

with initial condit^n as:

$$\frac{dy}{dx} = f(x, y), \; y(x_0) = y_0$$

We've to find an approx. value of (y) for $x_n = x_0 + nh$ starting with (x_0, y_0). where (h) is the step length i.e. $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, ---, $x_n = x_0 + nh$.

Step-1

Find the value of $y_1 = y(x_1)$, $y_2 = y(x_2)$ & $y_3 = y(x_3)$ by using any 1 of the methods previously discussed (Euler's methods or RK methods) if it isn't given.

Step 2

Find $y_1' = f_1 = f(x_1, y_1)$

$y_2' = f_2 = f(x_2, y_2)$

$y_3' = f_3 = f(x_3, y_3)$

Step 3 (1st approx)

By Milne's predictor formula:

$$y_4 = y(x_4) = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$$

Then, find $y_4' = f_4 = f(x_4, y_4)$

Step 4

By Milne's corrector formula:

$$y_4^* = y(x_4) = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$$

Then find $y_4^{*\prime} = f(x_4, y_4^*)$

Continuing this process for other approx. as well, we can get the results in general as:

ⓐ Milne's Predictor formula:

$$\boxed{y_{n+1} = y(x_{n+1}) = y_{n-3} + \frac{4h}{3}[2y_{n-2}' - y_{n-1}' + 2y_n']}$$

ⓑ Milne's corrector formula:

$$\boxed{y^*_{n+1} = y(x_{n+1}) = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right]}$$

Q: Using Milne's method, find $y(0.4)$ & $y(0.5)$ for the ODE $\frac{dy}{dx} = x - y^2$, $y(0) = 1$, taking $h = 0.1$, where $y_1 = y(0.1) = 0.9138$

$y_2 = y(0.2) = 0.85112$

$y_3 = y(0.3) = 0.8076$

Ans- Given,

$x_0 = 0$ $x_1 = 0.1$ $y_1 = 0.9138$

$y_0 = 1$ $x_2 = 0.2$ $y_2 = 0.85112$

$h = 0.1$ $x_3 = 0.3$ $y_3 = 0.8076$

$x_4 = 0.4$ $x_5 = 0.5$ $y_4 = ?$ $y_5 = ?$

$\therefore y_1' = f(x_1, y_1) = -0.735$

$y_2' = f(x_2, y_2) = -0.5244$

$y_3' = f(x_3, y_3) = -0.3522$

⊛ 1st approx:

(P) $y_4 = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right] = 0.78$

$\therefore y_4' = -0.2084 = f(x_4, y_4)$

(C) $y_4^* = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right] = \underline{0.77973}$

$\therefore y_4^{*\prime} = -0.207978 = f(x_4, y_4^*)$

⊛ 2nd approx:

(P) $y_5 = y_1 + \frac{4h}{3}\left[2y_2' - y_3' + 2y_4^{*\prime}\right] = 0.76545$

$\therefore y_5' = -0.08591 = f(x_5, y_5)$

(C) $y_5^* = y_3 + \frac{h}{2}\left[y_3' + 4y_4^{*\prime} + y_5'\right] = 0.765265$

So, $y_4 = \underline{0.77973}$ & $y_5 = 0.765265$

ii→ **Adam's method:-**

There are 2 types of Adam's method, the

explicit & the implicit types.

The explicit type is called the Adam's-Bashforth (AB) method & the implicit type is called the Adam's-Moulton (AM) method.

Let's consider an IVP (Initial Value Problem):

$$y' = \frac{dy}{dx} = f(x, y), \text{ such that } y(x_0) = y_0 \quad \text{---} ①$$

We've to find an approx. value of y for $x_n = x_0 + nh$, starting with (x_0, y_0).

Basically, Adam's-Bashforth is the predictor formula & Adam's-Moulton is the corrector formula.

To find the solutⁿ of equn. ① use:

⊛ Adam's-Bashforth predictor formula

$$\boxed{y_{n+1} = y_n + \frac{h}{24}\left[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}\right]}$$

Then find, $y'_{n+1} = f(x_{n+1}, y_{n+1})$

⊛ Adam's-Moulton corrector formula

$$\boxed{y^*_{n+1} = y_n + \frac{h}{24}\left[9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}\right]}$$

Then find $y^{*'}_{n+1} = f(x_{n+1}, y^*_{n+1})$ (If needed)

Continuing this procedure, we can solve an ordinary differential equn. by Adam's method.

N.B:

Adam's-Bashforth formula is an explicit formula. Hence, it doesn't involve y_{n+1} on its right.

Adam's-Moulton formula is an implicit formula because y_{n+1} appears on the right.

Q: Solve $\frac{dy}{dx} = 2e^x - y$, where:

$y_0 = 2$ $y_1 = 2.010$ $y_2 = 2.040$ $y_3 = 2.090$
$x_0 = 0$ $x_1 = 0.1$ $x_2 = 0.2$ $x_3 = 0.3$
$y_4 = ?$ $y_5 = ?$ $x_4 = 0.4$ $x_5 = 0.5$ $h = 0.1$

Ans- $y_0' = f(x_0, y_0) = 0$
$y_1' = f(x_1, y_1) = 0.2$
$y_2' = f(x_2, y_2) = 0.4$
$y_3' = f(x_3, y_3) = 0.6$

⊛ 1st approx

(P) $y_4 = y_3 + \frac{h}{24}[55y_3' - 59y_2' + 37y_1' - 9y_0']$
$= 2.16$

$\therefore y_4' = f(x_4, y_4) = 0.824$

(C) $y_4^* = y_3 + \frac{h}{24}[9y_4' + 19y_3' - 5y_2' + y_1']$
$= \underline{2.161}$ $\therefore y_4^{*\prime} = f(x_4, y_4^*) = 0.823$

⊛ 2nd approx

(P) $y_5 = y_4^* + \frac{h}{24}[55y_4^{*\prime} - 59y_3' + 37y_2' - 9y_1']$
$= 2.256$

$\therefore y_5' = f(x_5, y_5) = 1.04$

(C) $y_5^* = y_4^* + \frac{h}{24}[9y_5' + 19y_4^{*\prime} + 5y_3' + y_2']$

$= \underline{2.27932}$

So, $y_4 = \underline{2.161}$ & $y_5 = \underline{2.279}$